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Theory of vortex sound with special reference to vortex rings

Tsutomu Kambe
Higashi-yama 2-11-3, Meguro-ku, Tokyo 153-0043, Japan
kambe@gate01.com

From the fundamental conservation equations of mass, momentum and energy for an ideal fluid of uniform entropy, one can derive a wave equation of *aerodynamic sound*, which can be reduced to a compact form, called the wave equation of *vortex sound* with a source term of the form $\rho_0 \operatorname{div}(\boldsymbol{\omega} \times \boldsymbol{v})$ for the acoustic pressure p', where ρ_0 is the undisturbed density. We can consider generation of acoustic waves by vortex-vortex interactions on the basis of this equation. As an axisymmetric problem, we can deduce an expression of the wave profile generated by head-on collision of two coaxial vortex rings, which is characterized as *qudrupolar emission*.

On the other hand, based on the matched asymptotic expansion equivalent to the multipole expansions, one can derive a formula of wave pressure excited by time-dependent vorticity field localized in space. It can be applied to various problems of vortex sound such as an interaction of a vortex-ring with a solid body, and an oblique collision of two vortex rings as well as the head-on collision problem where the same formula as before can be derived. In the case of vortex-body interaction, it is found that the acoustic wave profile can be related to the time derivative of volume flux (through the vortex ring) of an imaginary potential flow around the body (as if like the Faraday's law), which is characterized as *dipolar emission*. The case of interaction of a vortex and a sharp edge is studied in terms of the original equation of aerodynamic sound becasue an edge plate is usually a non-compact body.

All the wave emissions were detected in the experiments of the author's group.

1 Introduction

It would be no exaggeration to say that any vortex motion excites acoustic waves. Physical idea is as follows. Suppose that there exists unsteady fluid flow whose vorticity distribution is compact, that is the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ is localized in space, *i.e.* $\boldsymbol{\omega}(\boldsymbol{x})$ is non-zero only for $\boldsymbol{x} \in D$, where D is a bounded domain in $\mathbb{R}^3$ with its length scale denoted by l. The vorticity field $\boldsymbol{\omega}$ induces a velocity field $\boldsymbol{v}(\boldsymbol{x})$ whose representative magnitude is denoted by u, where the velocity field is assumed to satisfy $\operatorname{div} \boldsymbol{v} = 0$ approximately, *i.e.*

the flow is incompressible approximately. The flow may be called a *vortex motion*. Thus, the vortex motion drives acoustic waves.

The flow field is surrounded by an outer wave field scaled on the length $\lambda = c\tau$, where c is the sound speed in the undisturbed medium at rest and $\tau = l/u$ is a typical time scale of the vortex motion. A Mach number of the flow may be defined by $M = u/c$, which is assumed to be much less than unity:

$$M = u/c \ll 1.$$

Owing to this condition, the whole space is separated into two: a space of *inner* source flow and that of *outer* field of wave propagation, because the wave scale $\lambda = l/M$ is much larger than the scale l of the source flow. Theory of vortex sound can be developed under these circumstances.

Reformulating the theory of aerodynamic sound by Lighthill [**1**], the sound source was identified with a term of the form $\rho_0 \operatorname{div}(\boldsymbol{\omega} \times \boldsymbol{v})$ at low M, first by Powell [**2**] and later by Howe [**3**], where ρ_0 is the undisturbed fluid density. Thus, the wave equation for the acoustic pressure p is written as

$$c^{-2}\,\partial_t^2 p - \nabla^2 p = \rho_0 \operatorname{div}(\boldsymbol{\omega} \times \boldsymbol{v}), \qquad (\partial_t = \partial/\partial t) \tag{1}$$

in the limit of $M(= u/c) \to 0$. This can be derived as an approximation from the conservation equations of mass, momentum and energy, which will be shown below.

It is assumed that the undisturbed state is characterized by uniform density ρ_0, pressure p_0, entropy s_0 and enthalpy w_0, and that ρ', p', s' and w' denote deviations from the uniform values. The shear viscosity μ, kinematic viscosity $\nu = \mu/\rho_0$, thermal conductivity k and sound speed $c = \sqrt{(\partial p/\partial \rho)_s}\,\big|_{\rho=\rho_0, p=p_0}$ are assumed constant.

2 Wave equations

2.1 Lighthill's equation of aerodynamic sound

We consider first the fundamental conservation equations of mass and momentum of a viscous fluid, which are written as

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0\,, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_k}\Pi_{ik} = 0, \tag{3}$$

$$\Pi_{ik} = \rho v_i v_k + (p - p_0)\delta_{ik} - \sigma_{ik}, \tag{4}$$

where Π_{ik} is the stress tensor and σ_{ij} is the viscous stress tensor defined by

$$\sigma_{ik} = \mu\, e_{ik}, \qquad e_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i - \tfrac{2}{3}\,\delta_{ik}\,\partial v_l/\partial x_l, \tag{5}$$

with μ the shear viscosity. A constant term $p_0\delta_{ik}$ is added to Π_{ik}, which is introduced for convenience and has no influence in the momentum equation.

Differentiating (2) with respect to t and taking divergence of (3), we can eliminate the common term of the form $\partial_t \partial(\rho v_i)/\partial x_i$ from the two equations. Thus, we obtain the following *Lighthill's equation* for ρ':

$$(\partial_t^2 - c^2\nabla^2)\,\rho' = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\,T_{ik}, \tag{6}$$

$(\rho' = \rho - \rho_0,\ p' = p - p_0)$, where T_{ik} is the Lighthill's tensor defined by

$$T_{ik} = \Pi_{ik} - c^2\,\rho'\,\delta_{ik} = \rho v_i v_k + (\,p' - c^2\,\rho'\,)\,\delta_{ik} - \sigma_{ik}. \tag{7}$$

The second term $c^2\nabla^2\rho'$ of (6) is newly added, which is to be cancelled by the term $c^2\rho'\delta_{ik}$ in T_{ik}.

Lighthill (1952) used the wave equation (6) to predict the well-known U^8-law, which means that the acoustic power emitted by a turbulent flow of representative flow speed U is proportional to U^8.

2.2 Reformulation of the equation of aerodynamic sound

Conservation equations of mass, momentum and energy of a viscous fluid can be rewritten as follows (Kambe & Minota [**4**, Appendix A]):

$$\nabla\cdot\boldsymbol{v} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\,, \tag{8}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{L} + \nabla(\tfrac{1}{2}\,\boldsymbol{v}^2) = -\frac{1}{\rho}\nabla\,p + \nu\nabla\cdot e, \tag{9}$$

$$T\frac{\mathrm{D}s}{\mathrm{D}t} = \nu\,e : \nabla\boldsymbol{v} + k\nabla^2 T, \tag{10}$$

where T and s are the temperature and entropy (per unit mass) respectively, $\mathrm{D}/\mathrm{D}t$ is the convective derivative $\partial_t + \boldsymbol{v}\cdot\nabla$, and

$$\boldsymbol{L} = \boldsymbol{\omega}\times\boldsymbol{v} \tag{11}$$

$$e \equiv (e_{ik}), \qquad \nabla\cdot e = (\partial/\partial x_k)e_{ik}, \qquad e : \nabla\boldsymbol{v} = e_{ik}\,(\partial/\partial x_i)v_k.$$

Using thermodynamic relations, the pressure p and density ρ can be expressed in terms of the entropy s and enthalpy w:

$$\frac{1}{\rho}\mathrm{d}p = \mathrm{d}w - T\,\mathrm{d}s, \qquad \frac{1}{\rho}\mathrm{d}\rho = \frac{1}{c^2}\,\mathrm{d}w - \left(\frac{T}{c^2} + \frac{1}{c_p}\right)\,\mathrm{d}s, \tag{12}$$

where c_p is the specific heat at constant pressure. On eliminating $\rho^{-1}\mathrm{grad}\,p$ by using the first equation of (12), the equation (9) is transformed to

$$\nabla(w + \tfrac{1}{2}\,v^2) + \frac{\partial\boldsymbol{v}}{\partial t} - T\nabla s = -\boldsymbol{L} + \nu\nabla\cdot e. \tag{13}$$

Similarly, with use of the second equation of (12), the equation (8) is written as

$$\nabla\cdot\boldsymbol{v} = -\frac{1}{c^2}\frac{\mathrm{D}}{\mathrm{D}t}w + \left(\frac{T}{c^2}+\frac{1}{c_p}\right)\frac{\mathrm{D}}{\mathrm{D}t}s\,. \tag{14}$$

Taking divergence of (13), we have

$$\nabla^2(w+\tfrac{1}{2}\,v^2) + \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{v} - \nabla\cdot(T\nabla s) = -\nabla\cdot\boldsymbol{L} + \nu\nabla\nabla : e, \tag{15}$$

where $\nabla\nabla : e = (\partial/\partial x_i)(\partial/\partial x_k)e_{ik} = \frac{4}{3}\,\nabla^2(\nabla\cdot\boldsymbol{v})$. The term $(\partial/\partial t)\nabla\cdot\boldsymbol{v}$ of (15) can be expressed in terms of w and s by using (14), in which $(\boldsymbol{v}\cdot\nabla)w$ can be eliminated by using the equation obtained with taking scalar product of $\boldsymbol{v}$ and (13). Thus, we find the following inhomogeneous wave equation:

$$(\nabla^2 - c^{-2}\,\partial_t^2)\;(w+\tfrac{1}{2}\,v^2) \;=\; -F(\boldsymbol{x},t), \tag{16}$$

$$\begin{aligned} F(\boldsymbol{x},t) = \;& \nabla\cdot\boldsymbol{L} + c^{-2}\,\partial_t^2 v^2 + c^{-2}\,\partial_t[(\boldsymbol{v}\cdot\nabla)\tfrac{1}{2}\,v^2] + \frac{\partial}{\partial t}\left(\left(\frac{T}{c^2}+\frac{1}{c_p}\right)\frac{\mathrm{D}s}{\mathrm{D}t}\right) \\ & +\nabla\cdot(T\nabla s) - \nu c^{-2}\partial_t[\boldsymbol{v}\nabla : e] + \tfrac{4}{3}\,\nu\nabla^2(\nabla\cdot\boldsymbol{v}). \end{aligned} \tag{17}$$

Obviously, the equation (16) is a wave equation for the wave variable $w+\frac{1}{2}\,v^2$ with a source term $F(\boldsymbol{x},t)$ defined by (17), and is called as the equation of *aerodynamic sound* of viscous flows.

This is transformed immediately to the following integral form (by the standard theory of wave equation):

$$\left[w+\tfrac{1}{2}\,v^2\right](\boldsymbol{x},t) = \frac{1}{4\pi}\int\frac{F(\boldsymbol{y},t_{\mathrm{r}})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}^3\boldsymbol{y}, \tag{18}$$

by using the retarded time $t_{\mathrm{r}} = t - |\boldsymbol{x}-\boldsymbol{y}|/c$, expressing the time delay due to the wave propagation from the source position at $\boldsymbol{y}$ to the observation station at $\boldsymbol{x}$.

2.3 Equation of vortex sound

Suppose that the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ is localized in space, *i.e.* the vorticity distribution is compact. Then the velocity field $\boldsymbol{v}(\boldsymbol{x})$ induced by the vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ has an asymptotic property decaying as $O(r^{-3})$ in the far field as $r = |\boldsymbol{x}| \to \infty$. In the far field where deviations from the uniform state is infinitesimal and the wave propagation is regarded as adiabatic (*i.e.* $\mathrm{d}s = 0$), we have $w+\frac{1}{2}\,v^2 \to p/\rho_0$ from the first of the thermodynamic relations (12) since $|\boldsymbol{v}(\boldsymbol{x})| = O(r^{-3})$. Then the equation (18) becomes

$$p(\boldsymbol{x},t) = \frac{\rho_0}{4\pi}\int\frac{F(\boldsymbol{y},t_{\mathrm{r}})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}^3\boldsymbol{y}, \qquad t_{\mathrm{r}} = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}. \tag{19}$$

It is well-known that the rate of viscous dissipation of kinetic energy, denoted by $K'(t)$, is given by

$$K'(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_V \tfrac{1}{2}\,v^2\,\mathrm{d}^3\boldsymbol{y} = -\nu\int_V e_{ik}\frac{\partial v_i}{\partial x_k}\,\mathrm{d}^3\boldsymbol{y}. \tag{20}$$

The heat delivered to the fluid in unit time is equal to the rate of dissipation of kinetic energy, *i.e.* $-K'(t)$ is equal to the space integral of $T\mathrm{D}s/\mathrm{D}t$. Using these relations, we can simplify the above expression (19) of acoustic pressure under the viscous action as follows (Kambe & Minota [**4**, Appendix A]):

$$p(\boldsymbol{x},t) = \frac{\rho_0}{4\pi}\int \frac{\nabla\cdot\boldsymbol{L}}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}^3\boldsymbol{y} + \frac{\rho_0}{4\pi c^2}(2-\gamma)\frac{1}{r}\,K''(t-\frac{r}{c}) \tag{21}$$

$$= -\frac{\rho_0}{4\pi c}\frac{x_i}{r^2}\frac{\partial}{\partial t}\int L_i(\boldsymbol{y},t-\frac{r}{c})\,\mathrm{d}^3\boldsymbol{y} + \frac{\rho_0}{4\pi c^2}(2-\gamma)\frac{1}{r}\,K''(t-\frac{r}{c}). \tag{22}$$

where $\boldsymbol{L} = \boldsymbol{\omega}\times\boldsymbol{v}$.

In an ideal fluid where there is no viscous dissipation, the kinetic energy K is constant. Hence the second term vanishes identically in the above expression (21) (or (22)). Then, the expression (21) implies the following differential equation:

$$\frac{1}{c^2}\,\partial_t^2 p - \nabla^2 p = \rho_0\,\nabla\cdot\boldsymbol{L}, \quad = \rho_0\,\mathrm{div}(\boldsymbol{\omega}\times\boldsymbol{v}), \tag{23}$$

which is called the *equation of vortex sound*. This is used as a basic governing equation in the present context, where it is assumed that the source vorticity $\boldsymbol{\omega}(\boldsymbol{x},t)$ is locallized and compact in space and its representative Mach number M is sufficiently low. It is remarked that $p(\boldsymbol{x},t)$ satisfies the wave equation (23) (approximately), when $\boldsymbol{x}$ is far from a *compact* source at $\boldsymbol{y}$. If the source term $\rho_0\,\mathrm{div}(\boldsymbol{\omega}\times\boldsymbol{v})$ is evaluated with the incompressible vortex motion, then the error would be $O(M^2)$.

The acoustic waves generated by vortex motion in an ideal fluid is also represented by the first term of (22) for a compact vorticity $\boldsymbol{\omega}$. The same expression can be derived from the other formulation, that is the matched asymptotic expansion equivalent to the multipole expansions, which is the subject of the next section.

3 Inner region and outer region

As described in the introduction, if the vorticity field is compact in space and the typical flow Mach number is much less than unity, the whole space is separated into two: a space of *inner* source flow and that of *outer* field of wave propagation, because the wave scale $\lambda = l/M$ is much larger than the scale l of the source flow.

3.1 Inner flow region

The *inner region* is scaled on l, the inner velocity is scaled on u, and inner dimensionless variables are denoted with a bar ($\tau = l/u$):

$$\bar{x}_i = \frac{x_i}{l}, \quad \bar{t} = \frac{t}{\tau}, \quad \bar{p} = \frac{p-p_0}{\rho_0 u^2}, \quad \bar{v}_i = \frac{v_i}{u}, \quad \bar{\nabla} = l\,\nabla. \tag{24}$$

Then, the equation (23) is rewritten as

$$\bar{\nabla}^2\bar{p} = -\,\overline{\mathrm{div}}(\bar{\boldsymbol{\omega}}\times\bar{\boldsymbol{v}}) + O(M^2). \tag{25}$$

The equation of an incompressible fluid is equivalent to neglecting the $O(M^2)$ terms.

Let us consider a solenoidal velocity field $\boldsymbol{v}(\boldsymbol{x})$ (*i.e.* $\mathrm{div}\,\boldsymbol{v} = 0$) induced by a compact vorticity field $\boldsymbol{\omega}(\boldsymbol{x})$, which is given at an initial instant in a bounded domain D_0 of linear dimension $O(l)$ and stays in bounded domain D at subsequent times. The $\boldsymbol{\omega}(\boldsymbol{x})$ is governed by the vorticity equation:

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \tag{26}$$

Introducing a vector potential $\boldsymbol{A}(\boldsymbol{x})$, the solenoidal velocity is expressed as

$$\boldsymbol{v}(\boldsymbol{x}, t) = \mathrm{curl}\,\boldsymbol{A}, \qquad \boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{\mathrm{D}} \frac{\boldsymbol{\omega}(\boldsymbol{y}, t)}{|\boldsymbol{x} - \boldsymbol{y}|} \,\mathrm{d}^3\boldsymbol{y}. \tag{27}$$

It can be shown that $\mathrm{curl}\,\boldsymbol{v} = \boldsymbol{\omega}$ in the free space.

At large distances $x \equiv |\boldsymbol{x}| \gg y \equiv |\boldsymbol{y}|$ for $\boldsymbol{y} \in \mathrm{D}$, we have an asymptotic expansion,

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} = \frac{1}{x} - y_i \frac{\partial}{\partial x_i} \frac{1}{x} + \tfrac{1}{2}\, y_i y_j \frac{\partial^2}{\partial x_i \,\partial x_j} \frac{1}{x} + \cdots . \tag{28}$$

Outside D, the velocity is irrotational (by definition) and represented by the form $\boldsymbol{v} = \mathrm{grad}\,\Phi$. The velocity potential Φ associated with the vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$ is given by the series expansion at large $x = |\boldsymbol{x}|$ as

$$\Phi(\boldsymbol{x}, t) = \frac{1}{4\pi} P_i \frac{\partial}{\partial x_i} \frac{1}{x} - Q_{ij} \frac{\partial^2}{\partial x_i \,\partial x_j} \frac{1}{x} + O(x^{-4}), \tag{29}$$

$$P_i = \tfrac{1}{2} \int_{\mathrm{D}} (\boldsymbol{y} \times \boldsymbol{\omega})_i \,\mathrm{d}^3\boldsymbol{y}, \qquad Q_{ij} = \frac{1}{12\pi} \int_{\mathrm{D}} y_i (\boldsymbol{y} \times \boldsymbol{\omega})_j \,\mathrm{d}^3\boldsymbol{y} \tag{30}$$

(see [**6**]). The term corresponding to the first term of (28) vanishes owing to the property $\int \omega_i \mathrm{d}^3\boldsymbol{y} = 0$. The vector P_i is the flow impulse, and the tensor Q_{ij} (satisfying $Q_{ii} = 0$) is a second moment of the vorticity distribution, which will be related later to the wave profile generated by the vortex motion. In general they depend on the time t. In the absence of external forces and bodies, the impulse P_i is conserved. It will be shown that the excitation of an acoustic wave by a rotational flow (27) is closely related to the time dependence of the coefficients of the multipole expansion (29). It is obvious from (29) that the magnitude of $v_i = \partial\Phi/\partial x_i$ is $O(x^{-3})$ as $x \to \infty$.

3.2 Outer wave region

Next, we consider the space with a much larger scale. Using the scaling length $\lambda (\gg l)$, we obtain the following estimate of magnitudes,

$$O\left(\frac{1}{c^2}\, p_{tt}\right) \Big/ O(\nabla^2 p) = \frac{\lambda^2}{c^2 \tau^2} \approx 1\,.$$

Hence, the two terms on the left hand side of (23) are comparable in magnitudes. In general, the pressure p associated with compressible motion (therefore p_{tt}/c^2 as well) decays as x^{-1}, whereas the velocity v_i on the right hand side decays like $O(x^{-3})$ for the solenoidal component.[1] Introducing the outer variables defined by

$$\hat{x}_i = \frac{x_i}{\lambda} = M\bar{x}_i, \qquad \hat{t} = \bar{t} = \frac{t}{\tau}, \qquad M = l/\lambda,$$
$$\hat{p} = \frac{p - p_0}{\rho_0 u^2}, \qquad \hat{v}_i = \frac{v_i}{u}, \qquad \hat{\nabla} = \lambda\,\nabla. \tag{31}$$

we find that the equation (23) is rewritten as

$$\frac{\partial^2}{\partial \hat{t}^2}\hat{p} - \hat{\nabla}^2 \hat{p} = 0, \tag{32}$$

neglecting $O(M^{4+2\beta})$ terms relative to those retained. This is a wave equation, implying that there exists a region of wave propagation at large distances.

4 Pressures in inner and outer regions

To the leading order, the flow in the inner region is governed by the equation of motion of an incompressible fluid, and the pressure of incompressible flows is determined by

$$\nabla^2 p = -\rho_0\,\nabla \cdot \boldsymbol{L}, \tag{33}$$

according to (25). This is a Poisson type equation for p. Introducing the Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ satisfying

$$\nabla^2_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = -\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{34}$$

where $\nabla_{\boldsymbol{x}}$ is a nabla operator with respect to the variable $\boldsymbol{x} = (x_i)$, we obtain the following integral representation for the inner pressure $p_{\rm I}$,

$$p_{\rm I} := p - p_0 = \rho_0 \int G(\boldsymbol{x}, \boldsymbol{y})\,\nabla \cdot \boldsymbol{L}(\boldsymbol{y}, t)\,\mathrm{d}^3\boldsymbol{y}. \tag{35}$$

in unbounded space, where $G(\boldsymbol{x}, \boldsymbol{y})$ is the free space Green's function given below.

If there is a solid body, boundary conditions are to be imposed on the body surface S:

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0, \qquad \text{on} \quad \mathrm{S}, \tag{36}$$
$$\boldsymbol{n} \cdot \nabla_{\boldsymbol{y}} G = 0, \qquad \text{for } \boldsymbol{y} \text{ on } \mathrm{S}, \tag{37}$$

where $\boldsymbol{n}$ is a unit normal to S. Then, the inner pressure is represented by

$$p_{\rm I}(\boldsymbol{x}, t) = -\rho_0 \int \boldsymbol{L}(\boldsymbol{y}, t) \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}. \tag{38}$$

[1] The compressible component of velocity, which is given by $p/c\rho_0$ in the linear theory of sound wave and infinitesimal, decays as $u(l/x)M^{1+\beta}$ $(\beta > 0)$.

It is remarkable that this integral representation is valid whether a solid body is present or not (Kambe [**5**]).

The outer region is governed by the wave quation (32). With using an arbitrary function $a(\hat{t})$ and $\hat{x} = |\hat{\boldsymbol{x}}|$, a function of the form $\hat{x}^{-1}\, a(\hat{t} - \hat{x})$ is a solution of the equation (32). Its derivative, obtained by differentiating arbitrary times with respect to the space coordinates $\hat{x_i}$, is a solution as well. Thus the acoustic pressure $p_{\rm O} = p - p_0$ in the outer region is represented in the form of multipole expansions:

$$p_{\rm O}(\boldsymbol{x}, t) = \frac{A_0(\hat{t} - \hat{x})}{\hat{x}} + \frac{\partial}{\partial \hat{x}_i} \frac{A_i(\hat{t} - \hat{x})}{\hat{x}} + \frac{\partial^2}{\partial \hat{x}_i\, \partial \hat{x}_j} \frac{A_{ij}(\hat{t} - \hat{x})}{\hat{x}} + \cdots , \qquad (39)$$

where $\hat{t} - \hat{x} = (t - x/c)/\tau$ is the retarded time in the outer variables. The functions $A_0(\hat{t})$, $A_i(\hat{t})$, $A_{ij}(\hat{t}), \cdots$ (with the dimension of pressure) are unknowns to be determined by matching to the inner pressure $p_{\rm I}$. In other words, the pressure $p_{\rm O}$ represents the acoustic waves generated by vortex motion if the functions $A_0(\hat{t})$, $A_i(\hat{t})$, $A_{ij}(\hat{t}), \cdots$ are expressed in terms of the vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$.

The matching of the two expressions $p_{\rm I}(\boldsymbol{x}, t)$ and $p_{\rm O}(\boldsymbol{x}, t)$ is carried out in an intermediate region, on the basis of the method of matched asymptotic expansions. By this matching, the functions $A_0(\hat{t})$, $A_i(\hat{t})$, $A_{ij}(\hat{t}), \cdots$ are represented with the vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$. Then, the wave $p_{\rm O}(\boldsymbol{x}, t)$ of (39) is called the *vortex sound*.

5 Vortex sound in free space

The Green's function in free space is given by

$$G(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|}. \qquad (40)$$

which is to be substituted in (38). In the present formulation, it is assumed that the point of observation $\boldsymbol{x}$ is at large distances from the point $\boldsymbol{y}$ located within the vortex flow, *i.e.* it is assumed that $x \gg y$ where $y = O(l)$. Using the expansion (28), one can find the following expansion [**5**],

$$\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi} \nabla \frac{1}{x} + \nabla_{\boldsymbol{y}} \times \boldsymbol{g} + O(x^{-4}), \qquad (41)$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{4\pi x^5} (\boldsymbol{x} \cdot \boldsymbol{y})(\boldsymbol{x} \times \boldsymbol{y}). \qquad (42)$$

When (41) is substituted into (38), it is readily seen that the contribution from the first term disappears. Regarding the second term, integration by parts transforms the integrand into the form $-\rho_0 (\nabla \times \boldsymbol{L}) \cdot \boldsymbol{g}$. Noting $\boldsymbol{L} = \boldsymbol{\omega} \times \boldsymbol{v}$, we find that $\nabla \times \boldsymbol{L} = -\partial_t \boldsymbol{\omega}$ from (26). Thus, the inner pressure is reduced to the expression,

$$\begin{aligned} p_{\rm I}(\boldsymbol{x}, t) &= \rho_0 \frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{\omega}(\boldsymbol{y}, t) \cdot \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}^3 \boldsymbol{y} + O(x^{-4}) \\ &= \rho_0\, Q_{ij}^{'}(t)\, \frac{\partial^2}{\partial x_i\, \partial x_j} \frac{1}{x} + O(x^{-4}), \end{aligned} \qquad (43)$$

where the tensor Q_{ij} is defined in (30) and the prime denotes differentiation with respect to the time t. It is remarkable that the factor $\partial^2/\partial x_i\,\partial x_j)x^{-1}$ of the leading term denotes quadrupole potentials with time-dependent coefficient $Q_{ij}^{'}(t)$.

The matching procedure described above leads to the expression of the outer pressure given by

$$p_{\rm O}(\boldsymbol{x},t) = \rho_0 \frac{\partial^2}{\partial x_i\,\partial x_j} \frac{Q_{ij}^{'}(t-x/c)}{x}. \tag{44}$$

This is the pressure formula of the vortex sound [**5**].

In the far field as $\hat{x} \to \infty$, the pressure takes the following simpler form,

$$p_{\rm F}(\boldsymbol{x},t) = \frac{\rho_0}{c^2} \frac{x_i\,x_j}{x^3} Q_{ij}^{'''}(t-x/c). \tag{45}$$

which has the property of a *qudupolar* wave, where from (30)

$$Q_{ij} = \frac{1}{12\pi} \int_{\rm D} y_i(\boldsymbol{y}\times\boldsymbol{\omega})_j\, {\rm d}^3\boldsymbol{y}. \tag{46}$$

6 Head-on collision of two vortex rings

Experimental detection of the vortex sound was made first for a head-on collision of two vortex rings [**4**]. This is an axisymmetric problem, in which vortex lines are circular with a common symmetry axis (taken as z-axis) and the vorticity has only the azimuthal component in the cylindrical coordinate system (z,R,ϕ), *i.e.* $\boldsymbol{\omega} = (0,\,0,\,\omega(z,R)\,)$. Using the spherical coordinates $\boldsymbol{x} = (r,\theta,\phi)$, the observation point $\boldsymbol{x} = (x,\,y,\,z)$ are expressed as $x = r\sin\theta\,\cos\phi$, $y = r\sin\theta\,\sin\phi$, and $z = r\cos\theta$. Then, the tensor Q_{ij} of (46) are found to be diagonal:

$$Q_{zz} = -2Q_{xx} = -2Q_{yy} := (1/6)\,Q(t), \qquad Q_{ij} = 0 \quad (i \neq j),$$

where

$$Q(t) := \int\!\!\int z\,R^2\,\omega(z,R,t)\ {\rm d}z\,{\rm d}R. \tag{47}$$

Thus, the far field acoustic pressure (45) generated by the head-on collision of two vortex rings is reduced to

$$p(r,\theta,t) = \frac{\rho_0}{4c^2}\frac{1}{r}\,(\cos^2\theta - \tfrac{1}{3})\ Q^{'''}(t-r/c), \tag{48}$$

where r, θ are the coordinates of the observation point. The azimuthal component of the vorticity $\omega(z,R,t)$ is governed by the vorticity equation (26), and determines the time evolution of the function $Q(t)$.

This formulation can be applied to a discrete set of N circular vortex rings whose common axis coincides with the z axis. The factor $\omega\,\mathrm{d}z\,\mathrm{d}R$ in the integrand of (47) stands for the strength ($\mathrm{d}\Gamma$, say) of an elemental vortex ring at (Z, R). Therefore the function $Q(t)$ for N vortex rings can be written as

$$Q(t) = \sum_{i=1}^{N} Z_i\, R_i^2\, \Gamma_i, \tag{49}$$

where R_i, Z_i and Γ_i are the radius, axial position and strength of the i-th vortex, respectively.

Head-on collision of two identical vortex rings ($i = 1,\ 2$) with opposite circulations is represented by $N = 2$, and we may set $R_1 = R_2 \equiv R(t)$, $Z_1 = -Z_2 \equiv Z(t)\,(> 0)$ and $-\Gamma_1 = \Gamma_2 \equiv \Gamma(> 0)$, where the mid-plane of collision is at $z = 0$. Thus, we may define

$$Q(t) = -2\,\Gamma R^2(t)\, Z(t). \tag{50}$$

The orbit $(Z(t),\ R(t))$ of a vortex ring in the inviscid fluid is given together with the time factor $Q'''(t)$ in [**7**].

A direct numerical simulation was carried out to obtain the time factor $Q'''(t)$ for the vortex collision in a viscous fluid, which is shown in [**8**].

7 Sound emission by vortex-body interaction

7.1 dipolar emission

When there is a solid body in the vicinity of the vortex motion, the wave field is characterized by a dipolar emission rather than the quadrupolar emission considered so far in free space. The boundary condition to be satisfied on the body surface causes the more powerful emission of waves of dipole nature. The changing pressure over the body surface results in changing net force acting on the body. Conversely the fluctuating force, multiplied by a minus sign, is equivalent to the rate of change of the resultant momentum of the fluid.

In the presence of a solid body of size $O(l)$ near the vortex motion, the Green's function is given approximately by

$$G_{\mathrm{B}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{Y}(\boldsymbol{y})|}, \tag{51}$$

$$\boldsymbol{Y}(\boldsymbol{y}) = \boldsymbol{y} + \Phi(\boldsymbol{y}). \tag{52}$$

This is valid when $\boldsymbol{x}$ is far from the body, *i.e.* $|\boldsymbol{x}| \gg l$. The vector function $Y_i(\boldsymbol{x})$ denotes the velocity potential (*i.e.* $\nabla^2 Y_i = 0$) of a hypothetical flow around the body with a unit velocity to the y_i direction at infinity ($i = 1, 2, 3$). The first term y_i represents the uniform flow of a unit velocity and the vector function $\Phi_i(\boldsymbol{y})$ represents a correction due to the presence of a body which imposes the boundary condition of vanishing normal velocity.

When there is no solid body, one may put $\Phi_i = 0$. Then the function (51) reduces to (40) for the free space.

The function $G_{\rm B}(\boldsymbol{x}-\boldsymbol{y})$ satisfies the boundary condition on the body surface S,

$$\boldsymbol{n}\cdot\nabla_{\boldsymbol{y}}\, G_{\rm B} = 0, \qquad \text{for } \boldsymbol{y} \text{ on S.}$$

This is verified by differentiating (51) with respect to y_i,

$$\nabla_{\boldsymbol{y}}\, G_{\rm B}(\boldsymbol{x},\boldsymbol{y}) = \frac{(x_i - Y_i)\,\nabla_{\boldsymbol{y}} Y_i}{4\pi|\boldsymbol{x}-\boldsymbol{Y}(\boldsymbol{y})|^3}\,, \tag{53}$$

and using the property $\boldsymbol{n}\cdot\nabla Y_i = 0$ for $\boldsymbol{y}$ on S, since ∇Y_i denotes the velocity of a potential flow around the body. It is readily seen that the function $G_{\rm B}$ tends to $1/[4\pi|\boldsymbol{x}-\boldsymbol{y}|]$ as $x/l,\ y/l \to \infty$, since we have $|\Phi(\boldsymbol{y})| = O(y^{-2})$ for a body of size $O(l)$.

For $|\boldsymbol{x}| \gg |\boldsymbol{y}|$ in (51), we develop it in a form similar to (28), but using Y_i in place of y_i and apply the operator $\nabla^2_{\boldsymbol{y}}$. Then, the first two terms disappear (since $\nabla^2 Y_i = 0$), and the third term gives

$$\nabla^2_{\boldsymbol{y}}\, G_{\rm B} = \frac{1}{8\pi}\nabla^2_{\boldsymbol{y}}(Y_i\, Y_j)\,\frac{\partial^2}{\partial x_i\,\partial x_j}\frac{1}{x} = O(x^{-3}).$$

Therefore the function $G_{\rm B}$ satisfies the equation $\nabla^2 G_{\rm B} = 0$ within an error of $O(x^{-3})$. However, the term of $\nabla_{\boldsymbol{y}}\, G_{\rm B}$ to be used in the following is a lower-order term of $O(x^{-2})$ (see below). Thus, $G_{\rm B}$ has the correct behavior up to that order. This permits us to use (51) as the Green's function in the present context.

Using an asymptotic expansion of the form (28), the expression (53) is written as

$$\nabla_{\boldsymbol{y}}\, G_{\rm B} = \frac{x_i}{4\pi x^3}\,\nabla_{\boldsymbol{y}} Y_i + O(x^{-3}). \tag{54}$$

The velocity field of the potential flow given by $\nabla_{\boldsymbol{y}} Y_i$ is solenoidal in the $\boldsymbol{y}$-space. This permits introduction of a vector potential $\boldsymbol{\Psi}_i(\boldsymbol{y})$ (a vector for each $i = 1, 2, 3$) by the relation,

$$\nabla_{\boldsymbol{y}} Y_i = \nabla_{\boldsymbol{y}} \times \boldsymbol{\Psi}_i, \qquad \mathrm{div}\,\boldsymbol{\Psi}_i = 0 \quad (i = 1, 2, 3). \tag{55}$$

In cases of two-dimension or axisymmetry, the vector $\boldsymbol{\Psi}_i$ is related to the stream function. Each component of $\boldsymbol{\Psi}_i$ is harmonic since $0 \equiv \nabla\times(\nabla\times\boldsymbol{\Psi}) = -\nabla^2\boldsymbol{\Psi}_i$. One may choose that $\boldsymbol{\Psi}_i = 0$ on S without violating (55). Using (54) and (55) in (38), one obtains

$$p_{\rm I}(\boldsymbol{x}, t) = -\frac{\rho_0}{4\pi}\frac{x_i}{x^3}\int \boldsymbol{L}\cdot(\nabla_{\boldsymbol{y}}\times\boldsymbol{\Psi}_i)\,\mathrm{d}^3\boldsymbol{y} + O(x^{-3}). \tag{56}$$

Integrating by parts and using (26), one finds

$$p_{\rm I}(\boldsymbol{x}, t) = -\frac{\rho_0}{4\pi}\,\dot{\Pi}_i\,\frac{\partial}{\partial x_i}\frac{1}{x} + O(x^{-3}), \tag{57}$$

$$\Pi_i(t) = \int \boldsymbol{\omega}(\boldsymbol{y}, t)\cdot\boldsymbol{\Psi}_i(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}. \tag{58}$$

Thus the inner pressure is of the form of a dipole potential in the leading order.

Corresponding outer pressure is given by

$$p_{\mathrm{O}}(\boldsymbol{x},t) = -\frac{\rho_0}{4\pi}\frac{\partial}{\partial x_i}\frac{\Pi_i(t-x/c)}{x}. \tag{59}$$

In the far field as $\hat{x} = x/\lambda \to \infty$, this reduces to

$$p_{\mathrm{F}}(\boldsymbol{x},t) = -\frac{\rho_0}{4\pi c}\,\dot{\Pi}_i(t-x/c)\,\frac{x_i}{x^2}. \tag{60}$$

This represents a dipolar emission by the vortex-body interaction.

7.2 Emission from a loop vortex : (Acoustic Faraday law)

The dipole-emission law (60) is given an interesting interpretation in the following way. Suppose that there exists a vortex tube forming a closed loop, of which the centerline is denoted by a closed curve C. Writing a line element of C as $\mathrm{d}\boldsymbol{s}$, $\Pi_i(t)$ of (58) is rewritten as

$$\Pi_i(t) = \Gamma \oint_{\mathrm{C}} \boldsymbol{\Psi}_i(\boldsymbol{s})\cdot \mathrm{d}\boldsymbol{s} = \Gamma \int_{\mathrm{S}} (\nabla_{\boldsymbol{y}} \times \boldsymbol{\Psi}_i(\boldsymbol{y}))\cdot \boldsymbol{n}\ \mathrm{d}S(\boldsymbol{y}) \tag{61}$$

where S is an open surface with the circumference bounded by the closed curve C, and $\nabla \times \boldsymbol{\Psi}_i$ represents the velocity of a hypothetical potential flow (with a unit velocity in the y_i direction) around the body. Thus, the second expression of Π_i represents the volume flux J_i of the hypothetical flow through the loop C multiplied by Γ:

$$\Pi_i(t) = \Gamma J_i(t), \qquad J_i(t) = \int_{\mathrm{S}} (\nabla \times \boldsymbol{\Psi}_i)\cdot \boldsymbol{n}\ \mathrm{d}S. \tag{62}$$

The volume flux J_i depends on the vortex position. Although the potential flow $\nabla \times \boldsymbol{\Psi}_i$ is steady, the flux J_i is time-dependent because the vortex position (represented by the curve C) changes.

Thus, the following law is found: when a vortex ring (not necessarily circular) moves near a solid body, the flux J_i through C changes with the time t, which causes sound emission according to (60):

$$p_{\mathrm{F}}(\boldsymbol{x},t) = -\frac{\rho_0\Gamma}{4\pi c}\,J_i''(t-\frac{x}{c})\,\frac{x_i}{x^2}. \tag{63}$$

This phenomenon is *analogous* to the Faraday's law in the electromagnetism. The present case of vortex sound is valid in an asymptotic sense. However, the Faraday's law in the electromagnetism is valid rigorously.

8 Vortex-Edge interaction

8.1 Pressure formula

When a vortex moves near by a sharp edge E of a semi-infinite plate (a non-compact body), the wave emission is quite different from a compact body.

The acoustic pressure is given by

$$p(\boldsymbol{x},t) = -\rho_0 \int \boldsymbol{L}(\boldsymbol{y},\tau)\cdot\nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y};t-\tau)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau.$$

There is an additional time integral because of the non-compactness of the body. The Green's function G satisfies the boundary ondition on the edge-plate E:

$$x_1 < 0, \quad x_2 = 0, \quad -\infty < x_3 < \infty.$$

Using an approximate Green's function G_* (Kambe 1986), we obtain

$$p(\boldsymbol{x},t) = \frac{\rho_0}{\sqrt{2\pi^3}}\,\frac{\sin\frac{1}{2}\theta\,(\sin\phi)^{\frac{1}{2}}}{\sqrt{c}\,x}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{3}{2}}\int \boldsymbol{\omega}(\boldsymbol{y},t)\cdot\boldsymbol{\Psi}(\boldsymbol{y})\;\mathrm{d}^3\boldsymbol{y}, \tag{64}$$

where $\phi = \cos^{-1}(x_3/x)$. The fractional derivative is defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{1}{2}} g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega)^{\frac{1}{2}}\;\hat{g}(\omega)\,e^{-i\omega t}\mathrm{d}\omega = \int_{-\infty}^{t}\dot{g}(s)\,\frac{\mathrm{d}s}{[\pi(t-s)]}\;.$$

The acoustic pressure is composed of the angular factor,

$$F(\theta,\,\phi) = \sin\tfrac{1}{2}\,\theta\,(\sin\phi)^{\frac{1}{2}}\,, \qquad \text{(Cardioid)}$$

and the time factor,

$$f(t) = \Gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{3}{2}} J(t), \qquad J(t) = \int_{\mathrm{S}}(\nabla\times\boldsymbol{\Psi})\cdot\boldsymbol{n}\;\mathrm{d}S.$$

The function $J(t)$ represents the volume flux J through S of the hypothetical potential flow flowing around the edge E (Kambe, 1986).

9 Oblique collision of two vortex rings

9.1 Pressure formula

A general higher order expansion formula was developed to represent acoustic emission by an oblique collision of two vortex rings in a viscous fluid [9]. It is found that a third order component is found to be significant.

The acoustic pressure in the far field is given by

$$p(\boldsymbol{x},t) = \frac{5-3\gamma}{12}\,\frac{\rho_0}{\pi c^2}\,\frac{1}{x}\,K^{(2)}(t_{\mathrm{r}}) + \frac{\rho_0}{c^2}\,\frac{x_i\,x_j}{x^3}\,Q^{(3)}_{ij}(t_{\mathrm{r}}) + \frac{\rho_0}{c^3}\,\frac{x_i\,x_j\,x_k}{x^3}\,Q^{(4)}_{ijk}(t_{\mathrm{r}}) + \cdots\,,$$

where $t_{\mathrm{r}} = t - x/c$,

$$Q_{ijk}(t) = \frac{1}{12\pi}\int_{\mathrm{D}}(\boldsymbol{y}\times\boldsymbol{\omega})_i\,y_j\,y_k\,\mathrm{d}^3\boldsymbol{y},$$

and the superscript (n) denotes the n-th time derivative.

The **third term** represents a deviation from the **axi-symmetry**, which vanishes in the case of head-on collision of two vortex rings. However, in the oblique collision, it is found that the asymmetric terms are significant. This was observed in the experiment by Kambe, Minota & Takaoka (1993).

10 Summary

General pressure formula has been presented in the following form:

$$\begin{aligned} p(\boldsymbol{x},t) \quad &= \quad \frac{5-3\gamma}{12}\frac{\rho_0}{\pi c^2}\frac{1}{x}K^{(2)}(t_\mathrm{r}) - \frac{\rho_0}{4\pi c}\frac{x_i}{x^2}P_i^{(2)}(t_\mathrm{r}) \\ &\quad + \frac{\rho_0}{c^2}\frac{x_i\,x_j}{x^3}Q_{ij}^{(3)}(t_\mathrm{r}) + \frac{\rho_0}{c^3}\frac{x_i\,x_j\,x_k}{x^3}Q_{ijk}^{(4)}(t_\mathrm{r}) + \cdots \ . \end{aligned}$$

Sound generations by vortex rings are reviewed for the following four cases:

A. Head-on collision of two vortex rings.

B. Vortex-Body interaction.

C. Vortex-Edge interaction.

D. Oblique collision of two vortex rings.

REFERENCES

1 Lighthill MJ. "On sound generated aerodynamically. I", *Proc. R. Soc. London A* **211**, 564 - 587, 1952.

2 Powell A. "Theory of vortex sound", *J. Acoust. Soc. Am.* **36**, 177-195, 1964.

3 Howe MS. "Contributions to the theory of aerodynamic sound", *J. Fluid Mech.* **71**, 625-973, 1975.

4 Kambe T and Minota T. "Acoustic wave radiated by head-on collision of two vortex rings", *Proc. R. Soc. Lond. A* **386**, pp.277-308, 1983.

5 Kambe T. "Acoustic emissions by vortex motions", *J. Fluid Mech.* **173**, 643-666, 1986.

6 Kambe T, Minota T and Takaoka M. "Oblique collision of two vortex rings and its acosutic emission", *Phys. Rev. E* **48**, 1866 - 1881, 1993.

7 Kambe T and Minota T. "Sound radiation from vortex systems", *J. Sound & Vib.* **74**, 61 - 72, 1981.

8 Inoue, O, Hattori, Y and Sasaki T. "Sound generation by coaxial collision of two vortex rings", *J. Fluid Mech.* **424**, 327-365, 2000.